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Stochastic Dynamics of a Simple Food Chain Model with Fear Effect

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ABSTRACT
This paper is concerned with a stochastic food chain model. The model involves meso-carnivore on top, its prey in the middle and the prey's prey in the bottom. Here growth rate of meso-carnivore is affected due to the fear of virtual large carnivore is influenced by the playback of the large carnivore. The deterministic model is already investigated and we state some of the important analytical findings here. First, the existence and uniqueness of the global positive solutions of the system is obtained. Second, the stochastically ultimate boundedness of the solutions is studied. Then, the conditions for permanence and extinction of prey and predator populations are derived. To examine weak stability of stochastic system, the existence of a unique stationary distribution is shown under some conditions. Analytical results are verified numerically.

KEYWORDS: Fear effect; food chain model; stochastic permanence; extinction; stationary distribution

I. INTRODUCTION

Predator-prey interactions received much attention to the researchers due to its universal existence. The predator influences on prey population in two ways. First one is the direct impact where predator attack prey population and consume them whereas in the second case, predator not only attack the prey population but also induce fear and such fear can alter the behavioral pattern of prey population. From literature survey it is known that reproduction and foraging behavior of preyis also affected by fear. Recently it is shown that in [11], fear has a role on the death rate of prey population. Experimental studies in [20], shows a decrease of song sparrows due to fear effect. Based on the experimental work [20], several models are formulated and investigated [9,10,11,12,13,19] by considering the fear effect in predator-prey interactions. Further, from field studies, it is observed that playback of predator sounds influence emotions of prey. For example, playback sound of dolphin affects the emotions of Gulf toad fish [14].

In ecosystem, multiple species occur as a result food chain appears. So a natural question arises how a fear effect generated by multiple species influences the whole food chain. Panday et al. [13] studied the effect of fear in a tri-trophic food chain model where the growth rate of middle predator is affected by fear. Experimental observation in [17], showed the fear of large carnivore on meso-carnivores. This study reveals that foraging behavior of meso-carnivores reduces which in turn benefit the meso-carnivore's prey. Motivated by these observations, Wang and Zou [18] investigated the Lotka-Volterra food chain model

$$\frac{dx_1}{d\tau} = x_1(r_1 - a_{11}x_1 - a_{12}x_2),$$
$$\frac{dx_2}{d\tau} = x_2(r_2 - a_{22}x_2 - a_{23}x_3 + a_{21}x_1),$$
$$\frac{dx_3}{d\tau} = x_3(B(\alpha) - D - a_{33}x_3 + a_{32}x_2). \quad (1)$$

Here, $x_i, i = 1, 2, 3$ are the densities of prey, middle predator and meso-carnivores respectively where growth rate of meso-carnivores is reduced by the large carnivore's playback. r_1, r_2 and $'B(\alpha) - D'$ denote the growth rate of each species respectively. The population of large carnivore is not considered in the model though their voices are played for justification [17]. $B(\alpha)$ measures the fear effect on meso-carnivore where αexhibits the meso-carnivore's anti-predator response level. $B(\alpha)$ obeys the following restrictions.

$$B^{/}(\alpha) < 0, B(0) = B_3 \text{ and } \lim_{\alpha\to\infty} B(\alpha) = 0.$$

Here D represents the mortality rate of meso-carnivore.$a_{ii}(i = 1,2,3)$denotes the intra-species competition. a_{12} and $a_{23}$are the predation rates. Conversion efficiency of two predators are denoted by $a_{21}$and a_{32} respectively. In [18], the authors investigated the role of meso-carnivore's anti-predator response levelα on the dynamics.

For mathematical simplicity, Wang and Zou [18] non-dimensionalized the model (1) with the scaling:$t = r_1\tau, x = \frac{a_{11}x_1}{r_1}, y = \frac{a_{12}x_2}{r_1}, z = \frac{a_{23}x_3}{r_1}$and then system (1) becomes

$$\frac{dx}{dt} = x(1 - x - y)$$
$$\frac{dy}{dt} = y(k - d_1y - z + b_1x)$$
$$\frac{dz}{dt} = z(f(\alpha) - d_2z + b_2y)(2)$$

Where

$$k = \frac{r_2}{r_1}, d_1 = \frac{a_{22}}{a_{12}}, d_2 = \frac{a_{33}}{a_{23}}, b_1 = \frac{a_{21}}{a_{11}}, b_2 = \frac{a_{32}}{a_{12}}, f(\alpha) = \frac{B(\alpha) - D}{r_1}, \quad f(0) = \frac{B_3 - D}{r_1}, \quad \lim_{\alpha\to\infty} f(\alpha) = -\frac{D}{r_1}.$$

From the thorough study in [18], we know that

1. The equilibrium point $E_0 = (0, 0, 0)$and $E_1 = (1, 0, 0)$ are always unstable.
2. If $1 - \frac{k}{d_1} < 0$ and $f(\alpha) < -\frac{kb_2}{d_1}$then $E_2 = (0, \frac{k}{d_1}, 0)$ is asymptotically stable.
3. If $k < d_1$and $f(\alpha) < -\frac{b_2(k+b_1)}{d_1+b_1}$then $E_{12} = (1 - \frac{k+b_1}{d_1+b_1}, \frac{k+b_1}{d_1+b_1}, 0)$ is asymptotically stable.
4. If $f(\alpha) > 0$ and $f(\alpha) > d_2(k + b_1)$ then $E_{13} = (1, 0, \frac{f(\alpha)}{d_2})$ is asymptotically stable.
5. If $d_2k > f(\alpha) > -\frac{kb_2}{d_1}$and $f(\alpha) > d_2k - d_1d_2 - b_2$then $E_{23} = (0, \frac{d_2k-f(\alpha)}{d_1d_2+b_2}, k - \frac{d_1(d_2k-f(\alpha))}{d_1d_2+b_2})$ is asymptotically stable.
6. If $d_2(k + b_1) > f(\alpha) > \max\,(d_2k - d_1d_2 - b_2, -\frac{b_2(k+b_1)}{d_1+b_1})$ then the positive equilibrium point $E^* = (x^*, y^*, z^*)$ where $x^* = 1 - y^*, y^* = \frac{d_2(k+b_1)-f(\alpha)}{d_2(d_1+b_1)+b_2}, z^* = k + b_1 - (d_1 + b_1)y^*$ is globally asymptotically stable.

In the real world, the environmental noise plays a crucial role in an ecological system. Most of the ecological systems do not obey the deterministic laws but fluctuate in random manner around some average values. So the population density cannot reach to the fixed value with the evolution of time. Due to continuous variations in the atmosphere the basic factors that govern the population growth as for example birth rate, death rate, immigration and immigration etc. may be altered non-deterministically. These observations motivated a lot of authors to introduce randomness in deterministic model to explore the impact of environmental variability, namely random noise in the differential equations and environmental fluctuations in the system parameters [1,4,5,6,7,8,15]. In [8], the authors addressed an important observation that the environmental noise can suppress a potential population explosion.

Several techniques are employed to consider the random effects in the model, both from biological and mathematical point of view [1]. Our analysis in this work is similar to that developed in [15]. The environmental noise usually considered as proportional to the variables. We assume that stochastic perturbations are of a white noise type which is directly proportional to $x(t), y(t), z(t)$ influenced on $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ in system (2).

In studying predator-prey interactions, permanence is a major part that ensures the survival of all population in future time. There are three types of definition are available in literature namely stochastic permanence, persistence in mean and almost sure permanence. Here we mainly study stochastic permanence by using Lyapunov function method.

The paper is structured as follows. In Section 2, we describe the stochastic model. The existence and uniqueness of positive solutions, boundedness, permanence and extinctionare are also discussed in the same section. The existence of unique stationary distribution is discussed in Section 3. We verify our results numerically in Section 4. Section 5 deals with a brief discussion.

II. THE STOCHASTIC MODEL

In our model (2), we include random effects to address some biological issues. We consider the stochastic perturbation as a white noise type that varies directly to $x(t), y(t)$ and $z(t)$. Thus we have the following stochastic differential equations with respect to system (1):

$dx = x(1-x-y)dt + \sigma_1 x dB_1(t),$
$dy = y(k - d_1 y - z + b_1 x)dt + \sigma_2 y dB_2(t),$
$dz = z(f(\alpha) - d_2 z + b_2 y)dt + \sigma_3 z dB_3(t)$ (3)

where $\sigma_i, i = 1,2,3$ are the intensity of noise and $B_i(t), i = 1,2,3$ are mutually independent Brownian motions.

2.1. Dynamical behavior of stochastic model

Let (Ω, F, P) be a complete probability space with a filtration $\{F_t\}, t \geq 0$ assuring the properties: right continuity and increasing whileF_0 contains all P- empty sets. Denote

$$U(t) = \big(x(t), y(t), z(t)\big) \text{ and the norm } |U(t)| = \sqrt{x^2(t) + y^2(t) + z^2(t)}.$$

We now discuss whether positive global solution of system (8) exists uniquely or not.

Theorem1. Given any initial data$\big(x(0), y(0), z(0)\big) \in \mathbb{R}_+^3$one can find a unique solution $U(t) = (x(t), y(t), z(t))$ of system (3) on $t \geq 0$ and will lie in $\mathbb{R}_+^3$with probability one.

Proof. Consider a function $V : \mathbb{R}_+^3 \to \mathbb{R}_+$defined by

$V(x, y, z) = x - 1 - \ln x + y - 1 - \ln y + z - 1 - \ln z$. Evidently, this function is non-negative. If $U(t) \in \mathbb{R}_+^3$, we get

$$LV(U) = (x-1)(1-x-y) + (y-1)(k - d_1 y - z + b_1 x) + (z-1)\{f(\alpha) - d_2 z + b_2 y\} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

$= (2-b_1)x - x^2 + (b_1 - 1)xy + (k + d_1 + 1)y - d_1 y^2 + (1 + f(\alpha) + d_2)z - d_2 z^2 - (1 - b_2)yz + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \leq M.$

Proceeding along the lines in [15], we can show our assertion.

Theorem 1 indicates that the solution of system (8) will lie in $\mathbb{R}_+^3$. Now we describe how the solutions changes in $\mathbb{R}_+^3$ thoroughly. Wedemonstrate the ultimate boundedness that concerns the biological validity of the model.

Definition 1. (see [15]) The solution $U(t) = \big(x(t), y(t), z(t)\big)$ of system (3) is known as stochastically ultimately bounded if for each $\varepsilon \in (0,1)$ there exists a τ=τ(ε) whenever$\big(x(0), y(0), z(0)\big) \in \mathbb{R}_+^3$, the solution $U(t)$ of system (3) satisfy the inequality

$\lim_{t\to\infty} \sup P\{|U(t)| > \tau\} < \varepsilon.$

Theorem 2. Given any initial data$\big(x(0), y(0), z(0)\big) \in \mathbb{R}_+^3$ system (3) is bounded.

Proof. Define the function $V_1 = e^t x^\theta, V_2 = e^t y^\theta$and $V_3 = e^t z^\theta$for $(x, y, z) \in \mathbb{R}_+^3$ and $0 < \theta < 1$.

With the help of Ito's formula, one obtain

$$dV_1 = LV_1 dt + \sigma_1 \theta e^t x^\theta dB_1(t), dV_2 = LV_2 dt + \sigma_2 \theta e^t y^\theta dB_2(t), dV_3 = LV_3 dt + \sigma_3 \theta e^t z^\theta dB_3(t)$$

where

$$LV_1 = e^t x^\theta \left\{1 + \theta(1-x-y) + \frac{\sigma_1^2 \theta(\theta-1)}{2}\right\},$$
$$LV_2 = e^t y^\theta \left\{1 + \theta(k - d_1 y - z + b_1 x) + \frac{\sigma_2^2 \theta(\theta-1)}{2}\right\},$$
$$LV_3 = e^t z^\theta \left[1 + \theta\{f(\alpha) - d_2 z + b_2 y\} + \frac{\sigma_3^2 \theta(\theta-1)}{2}\right].$$

Therefore, there existsconstants $M_i > 0, i = 1,2,3$such that $LV_i < M_i e^t, i = 1,2,3$. Consequently,
$e^t E x^\theta - E x(0)^\theta \leq M_1 e^t, e^t E y^\theta - E y(0)^\theta \leq M_2 e^t$,and $e^t E z^\theta - E z(0)^\theta \leq M_3 e^t$. Thus we have

$$\limsup_{t\to\infty} E x^\theta \leq M_1 < \infty, \limsup_{t\to\infty} E y^\theta \leq M_2 < \infty, \limsup_{t\to\infty} E z^\theta \leq M_3 < \infty.$$

Observe that

$$|U(t)|^\theta = (x^2(t) + y^2(t) + z^2(t))^{\theta/2} \leq 3^{\theta/2}\max\{x^\theta(t), y^\theta(t), z^\theta(t)\} \leq 3^{\theta/2}(x^\theta + y^\theta + z^\theta).$$

This implies that

$$\limsup_{t\to\infty} E|U(t)|^\theta \leq 3^{\theta/2}(M_1 + M_2 + M_3) < \infty.$$

Using the Chebyshev inequality, one can establish the required contention.

We have already shown existence, uniqueness and boundedness of system (3). These properties are useful but not sufficient. Still one more property namely permanence is highly preferable as it establishes the species remains in future time. The definition of permanence is stated below.

Definition 2. (see [15]) The solution $U(t) = (x(t), y(t), z(t))$ of system (3) are known to be stochastically permanent given any $\varepsilon \epsilon (0,1)$, there exists constants $\rho = \rho(\varepsilon) > 0$ and $\xi = \xi(\varepsilon) > 0$whenever $\big(x(0), y(0), z(0)\big) \in \mathbb{R}_+^3$, the solution $U(t)$ of system (3) satisfy both the inequalities

$\lim_{t\to\infty} \inf P\{|U(t)| \leq \rho\} \geq 1-\varepsilon$,

$\lim_{t\to\infty} \inf P\{|U(t)| \geq \xi\} \geq 1-\varepsilon$.

Before stating our theorem on permanence, we require the following hypothesis:

$$H_1 : (\max\{\sigma_1, \sigma_2, \sigma_3\})^2 < 2\min\{1, k, f(\alpha)\}$$

Theorem 3. Given any initial data $\big(x(0), y(0), z(0)\big) \in \mathbb{R}_+^3$, the solution $(x(t), y(t), z(t))$ of system (3) satisfies

$$\lim_{t\to\infty} \sup E\left(\frac{1}{|U(t)|^\eta}\right) \leq \frac{3^\eta H}{\omega}$$

where H is a positive constant and η and ω are arbitrary positive constants satisfying

$$\eta\min\{1, \mathrm{k}, \mathrm{f}(\alpha)\} > \frac{\eta(1+\eta)}{2}(\max\{\sigma_1, \sigma_2, \sigma_3\})^2 + \omega \qquad (4)$$

Proof. Let $Q(x, y, z) = \frac{1}{x+y+z}$ for $\big(x(t), y(t), z(t)\big) \in \mathbb{R}_+^3$. Applying Ito's formula, we get

$$\begin{aligned} dQ &= -Q^2\{x - x^2 + (b_1 - 1)xy + (b_2 - 1)yz + ky - d_1y^2 + f(\alpha)z - d_2z^2\}dt + Q^3(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2) \\ &\quad - Q^2\big(\sigma_1 x dB_1(t) + \sigma_2 y dB_2(t) + \sigma_3 z dB_3(t)\big) \\ &= LQdt - Q^2\big(\sigma_1 x dB_1(t) + \sigma_2 y dB_2(t) + \sigma_3 z dB_3(t)\big). \end{aligned}$$

Choose a positive constant η satisfying (4) and applying Ito's formula, we have

$$\begin{aligned} &L(1+Q)^\eta \\ &= \eta(1+Q)^{\eta-1}LQ + \frac{\eta(\eta-1)}{2}Q^4(1+Q)^{\eta-2}(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2) \\ &= (1+Q)^{\eta-2}\psi \end{aligned}$$

where

$$\begin{aligned} \psi &= -\eta Q^2\{x - x^2 + (b_1 - 1)xy + (b_2 - 1)yz + ky - d_1y^2 + f(\alpha)z - d_2z^2\} \\ &\quad - \eta Q^3\{x - x^2 + (b_1 - 1)xy + (b_2 - 1)yz + ky - d_1y^2 + f(\alpha)z - d_2z^2\} \\ &\quad + \eta Q^3(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2) + \frac{\eta(\eta+1)}{2}Q^4(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2) \\ &\leq -\eta Q^2\{x + ky + f(\alpha)z - xy - yz - x^2 - d_1y^2 - d_2z^2\} \\ &\quad - \eta Q^3(x + ky + f(\alpha)z - xy - yz - x^2 - d_1y^2 - d_2z^2) + \eta Q^3(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2) \\ &\quad + \frac{\eta(\eta+1)}{2}Q^4(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2) \end{aligned}$$

Now

$$\begin{aligned} Q^3(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2) &< (\max\{\sigma_1, \sigma_2, \sigma_3\})^2 Q, \\ Q^4(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2) &< (\max\{\sigma_1, \sigma_2, \sigma_3\})^2 Q^2 \end{aligned}$$

So,

$\psi \leq$
$-Q^2[\eta\min\{1, k, f(\alpha)\} - \frac{\eta(1+\eta)}{2}(\max\{\sigma_1, \sigma_2, \sigma_3\})^2] + Q[\eta\max\{1, 1, 1, d_1, d_2\} -$
$\eta\min\{1, \mathrm{k.}\ \mathrm{f}(\alpha)\} + \eta(\max\{\sigma_1, \sigma_2, \sigma_3\})^2] + \max\{1, 1, 1, d_1, d_2\} + \eta(\max\{\sigma_1, \sigma_2, \sigma_3\})^2$.

Now, let $\omega > 0$ is small enough so that the inequality (4) is fulfilled. Clearly, we have

$$\begin{aligned} &L[e^{\omega t}(1+Q)^\eta] \\ &= \omega e^{\omega t}(1+Q)^\eta + e^{\omega t}L(1+Q)^\eta \\ &= e^{\omega t}(1+Q)^{\eta-2}\{\omega(1+Q)^2 + \psi\} \\ &\leq e^{\omega t}(1+Q)^{\eta-2}[\omega + \eta\max\{1, 1, 1, d_1, d_2\} + (2\omega + \eta\max\{1, 1, 1, d_1, d_2\} \\ &\quad - \eta\min\{1, \mathrm{k.}\ \mathrm{f}(\alpha)\} + \eta(\max\{\sigma_1, \sigma_2, \sigma_3\})^2)Q - (\eta\min\{1, \mathrm{k.}\ \mathrm{f}(\alpha)\} - \frac{\eta(\eta+1)}{2}(\max\{\sigma_1, \sigma_2, \sigma_3\})^2 \\ &\quad - \omega)Q^2] \end{aligned}$$

There exists a positive constant H such that

$$L[e^{\omega t}(1+Q)^\eta] \leq He^{\omega t}$$

Then $E[e^{\omega t}(1+Q)^\eta] \leq \left(1+Q(0)\right)^\eta + \frac{H}{\omega}(e^{\omega t}-1)$

So we can obtain $\lim_{t\to\infty}\sup E[Q(t)^\eta] \leq \lim_{t\to\infty}\sup E[(1+Q)^\eta] \leq \frac{H}{\omega}$.

For $(x,y,z) \in \mathbb{R}_+^3$, we know that $(x+y+z)^\eta \leq 3^\eta(x^2+y^2+z^2)^{\eta/2} \leq 3^\eta|U(t)|^\eta$; which in turn implies $\lim_{t\to\infty}\sup E\left(\frac{1}{|U(t)|^\eta}\right) \leq 3^\eta \lim_{t\to\infty}\sup E(Q(t)^\eta) \leq \frac{3^\eta H}{\omega}$

which completes the proof.

Theorem 4. Let the hypothesis H_1 be satisfied. Then system (3) is stochastically permanent.

Proof. The proof can be completed from Theorems 2 and 3 and application of Chebyshev inequality. So it is deleted here.

We have already shown that under certain restrictions, system (3) is ultimately bounded and stochastically permanent. Now, we show that if the noise is increased, system (3) may not be permanent i.e., environmental noise make the population extinct.

Theorem 5. Suppose that $2<\sigma_1^2, 2(k+b_1)<\sigma_2^2$and $2(f(\alpha)+\frac{kb_2}{d_1})<\sigma_3^2$. Then for any initial data $\left(x(0),y(0),z(0)\right) \in \mathbb{R}_+^3$, the solution $(x(t),y(t),z(t))$ of system (3) extinct with probability one.

Proof. From Ito's formula, we have $\ln x(t) \leq \left(1-\frac{\sigma_1^2}{2}\right)t+\sigma_1 B_1(t)+\ln x(0)$,

$$\ln y(t) \leq \left(k+b_1-\frac{\sigma_2^2}{2}\right)t+\sigma_2 B_2(t)+\ln y(0),$$

$$\ln z(t) \leq \left(f(\alpha)+\frac{kb_2}{d_1}-\frac{\sigma_1^2}{2}\right)t+\sigma_3 B_3(t)+\ln z(0).$$

Based on the strong law of large number of martingales we get $\lim_{t\to\infty}[\frac{\sigma_i B_i(t)}{t}+\frac{\ln_{i0}}{t}]=0, i=1,2,3$.

And by the assumptions of the theorem, we get $\lim\sup\frac{\ln x(t)}{t} \leq 1-\frac{\sigma_1^2}{2}<0$ a. s.,

$$\lim\sup\frac{\ln y(t)}{t} \leq k+b_1-\frac{\sigma_2^2}{2}<0 \text{ a. s.},$$

$$\lim\sup\frac{\ln z(t)}{t} \leq f(\alpha)+\frac{kb_2}{d_1}-\frac{\sigma_3^2}{2}<0 \text{ a. s.}$$

Therefore, system (3) is extinct exponentially. The proof is complete.

III. STATIONARY DISTRIBUTION

It is a known fact that stochastic models do not have interior equilibrium point when environmental noise is considered in the deterministic system. So, a stochastic weak stability called stationary distribution becomes an interesting part of research which has many applications in several discipline.

System (3) is now written in the form

$$d\begin{pmatrix}x(t)\\y(t)\\z(t)\end{pmatrix}=\begin{pmatrix}x(1-x-y)\\y(k-d_1y-z+b_1x\\z(f(\alpha)-d_2z+b_2y\end{pmatrix}dt+\begin{pmatrix}\sigma_1x\\0\\0\end{pmatrix}dB_1(t)+\begin{pmatrix}0\\\sigma_2y\\0\end{pmatrix}dB_2(t)+\begin{pmatrix}0\\0\\\sigma_3z\end{pmatrix}dB_3(t)$$

The corresponding diffusion matrix is given by

$$A(x,y,z)=\begin{pmatrix}\sigma_1^2x^2&0&0\\0&\sigma_2^2y^2&0\\0&0&\sigma_3^2z^2\end{pmatrix}.$$

Here, $X(t)=(x(t),y(t),z(t))^T$ is a homogeneous Markov process in E_3 (Euclidean 3-space). We show the existence of a stationary distribution for system (3). We mainly use the results developed in [3], and the technique used in [4]. It is to observe that there are a family of countable subsets $M_n$such that $\mathbb{R}_+^3=\cup_{n=1}^\infty M_n$. To establish the existence of a stationary distribution for system (3), we have to find a bounded domain$D \subseteq \mathbb{R}_+^3$ with regular boundary which fulfils the following assumptions.

Assumption A_1. (Page 118 of [3]) In the domain D and some neighborhood thereof, the least eigenvalue of the diffusion matrix $A(x,y,z)$ is bounded away from zero.

Assumption A_2. (Page 134 of [3]) $\text{Sup}_{M_n}E_x\tau<\infty$ for all n where $E_x\tau$ is the mean time τ at which a path issuing from x reaches the set M_n.

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Lemma 1. (See page 134 of [3]) Suppose the assumptions A_1 and A_2 hold. Then the Markov process $X(t)$ has a unique stationary distribution $\mu(\cdot)$ such that $\mu(E_3 \backslash \mathbb{R}_+^3) = 0$.

Theorem 6. Let $\sigma_i > 0, i = 1, 2, 3.$ Assume that $d_2(k + b_1) > f(\alpha) > \max\{d_2 k - d_1 d_2 - b_2, -\frac{b_2(k+b_1)}{d_1+b_1}\}$ and $\sigma < b_1 x^{*2} + d_1 y^{*2} + \frac{d_2}{b_2} z^{*2}$where $\sigma = \frac{\sigma_1^2 b_1}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2}$and (x^*, y^*, z^*) is the positive equilibrium point of system (2).
Then there is a stationary distribution $\mu(\cdot)$ with respect to $\mathbb{R}_+^3$ for system (3).
Proof. Define a function $V : \mathbb{R}_+^3 \to \mathbb{R}_+$ as

$$V(x, y, z) = b_1\left(x - x^* - x^* \ln \frac{x}{x^*}\right) + y - y^* - y^* \ln \frac{y}{y^*}) + \frac{1}{b_2}(z - z^* - z^* \ln \frac{z}{z^*}).$$

Evidently $V(x, y, z)$ is a positive definite function for all $(x, y, z) \neq (x^*, y^*, z^*)$. From Ito's formula, we have

$$\begin{aligned} LV(x, y, z) &= b_1(x - x^*)(1 - x - y) + (y - y^*)(k - d_1 y - z + b_1 x) + \frac{1}{b_2}(z - z^*)(f(\alpha) - d_2 z + b_2 y) \\ &\quad + \frac{\sigma_1^2 b_1 x^*}{2} + \frac{\sigma_2^2 y^*}{2} + \frac{\sigma_3^2 z^*}{2b_2} \\ &= -b_1(x - x^*)\{(x - x^*) + (y - y^*)\} - (y - y^*)\{d_1(y - y^*) + (z - z^*) - b_1(x - x^*)\} \\ &\quad - \frac{1}{b_2}(z - z^*)\{d_2(z - z^*) - b_2(y - y^*)\} + \frac{\sigma_1^2 b_1 x^*}{2} + \frac{\sigma_2^2 y^*}{2} + \frac{\sigma_3^2 z^*}{2b_2} \\ &= -b_1(x - x^*)^2 - d_1(y - y^*)^2 - \frac{d_2}{b_2}(z - z^*)^2 + \frac{\sigma_1^2 b_1 x^*}{2} + \frac{\sigma_2^2 y^*}{2} + \frac{\sigma_3^2 z^*}{2b_2} \triangleq -H(x, y, z) + \sigma \end{aligned}$$

Where $\begin{cases} H(x, y, z) = 0 \text{ if } x = x^*, y = y^*, z = z^* \\ H(x, y, z) > 0 \quad \text{otherwise} \end{cases}$

And $H(x, y, z) \to \infty$ as $x \to \infty$ or $y \to \infty$ or $z \to \infty$. By the continuity,we choose$D$ to be $\{(x, y, z): H(x, y, z) \leq \sigma\} \cap \mathbb{R}_+^3$such that for any $(x, y, z) \epsilon\, \mathbb{R}_+^3 \backslash D, LV$ is negative which ensures the assumption A_2. Again we see that if the condition () holds, there is a region D bounded away from zero. Hence there exists a constant K such that $\sigma_1^2 x^2 \delta_1^2 + \sigma_2^2 y^2 \delta_2^2 + \sigma_3^2 z^2 \delta_3^2 \geq |K||\delta|^2$ for all $(x, y, z)\, \epsilon\, D, \delta\, \epsilon\, \mathbb{R}^3$ which ensures the assumption A_1 (see Chapter 3 of [2]) and Rayleigh's principle in [16]. Therefore, system (3) has a stationary distribution $\mu(\cdot)$ which lie in $\mathbb{R}_+^3$.

IV. NUMERICAL STUDY

In order to understand of our analytical results in the stochastic system (3), we carry out numerical simulations using hypothetical set of parametric values. For this reason, we consider a particular function $f(\alpha)$ as $f(\alpha) = \frac{r}{1+c\alpha} - d$. We choose the parameters $k = 1, d_1 = 0.5, b_1 = 0.5, d_2 = 0.5, b_2 = 0.5, r = 3, c = 1, \alpha = 1, d = 1.25.$ (5)

Under different noise intensities σ_1^2, σ_2^2 and σ_3^2, we plot system (3) with initial point $\big(x(0), y(0), z(0)\big) = (0.6, 0.3, 0.1)$. In Fig. 1, without noise (i.e.,$\sigma_1^2 = 0, \sigma_2^2 = 0, \sigma_3^2 = 0$), the equilibrium point$(x^*, y^*, z^*) = (0.5, 0.5, 1)$ is globally asymptotically stable population distribution of three species. In Fig. 2, with white noise$\sigma_1^2 = 0.1, \sigma_2^2 = 0.16, \sigma_3^2 = 0.25$ satisfying the conditions of Theorems 4 and 6, one can observe that the random white noise generates a slight oscillation with the time independent stability and stationary distribution. Comparing Figures 1 and 2, one can understand that, if the intensities of white noise are low random fluctuations in the environment does not affect the dynamics so much. In this situation, we also observe stationary distribution of the system.Further from Fig. 2 and Fig. 3 we observe that the trajectories of the system oscillate randomly with remarkable variance of amplitude with the increasing values of the strength of white noises.Increasing the value of $\sigma_1^2, \sigma_2^2$and σ_3^2 to 4, 9, 16 respectively which satisfy the conditions of Theorem 5 and hence all the populations go to extinction. Figure 4 depicts that with the higher initial densities of populations, ultimately the entire population tend to extinct in the long run with the large intensities of noise. This indicates that population goes to extinction when the intensity of noise is large while the population can survive under relatively small white noise.

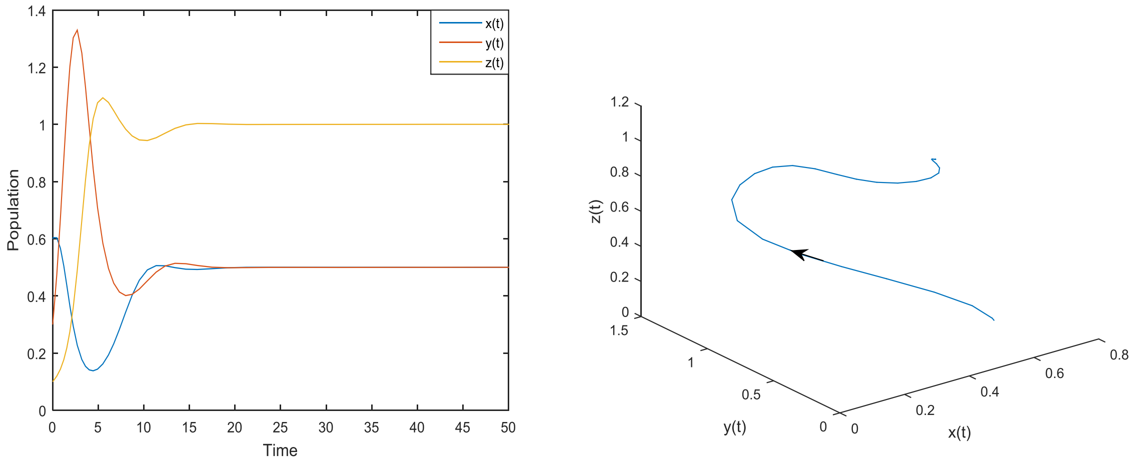

Fig.1.Time series plots along with phage portrait of system (3) with initial value $(x(0), y(0), z(0)) = (0.6, 0.3, 0.1)$and parameters are defined in (5) without noise.

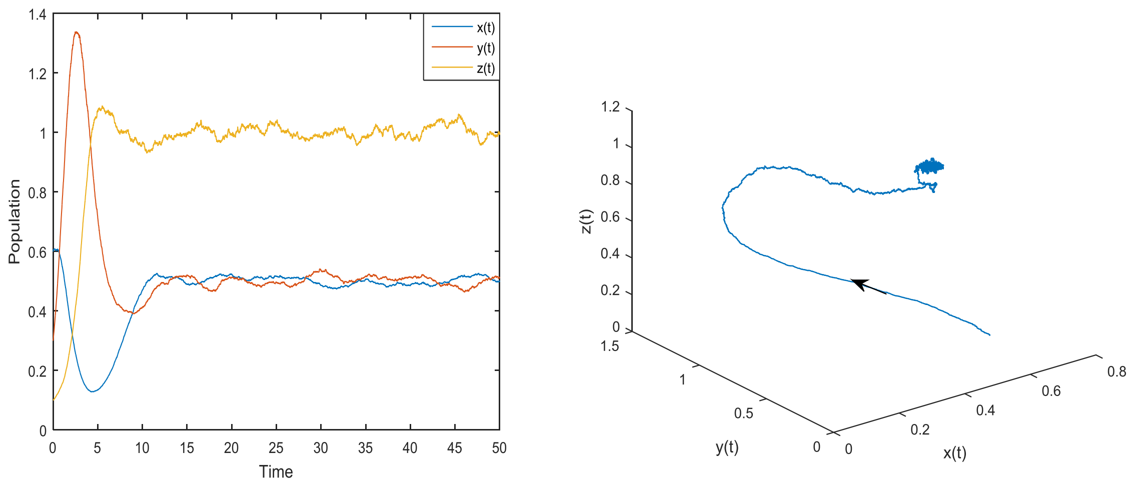

Fig.2. Time series plots along with phage portrait of system (3) with initial value $(x(0), y(0), z(0)) = (0.6, 0.3, 0.1)$and parameters are defined in (5) with σ_1=0.31623, σ_2 = 0.4, σ_3 =0.5.

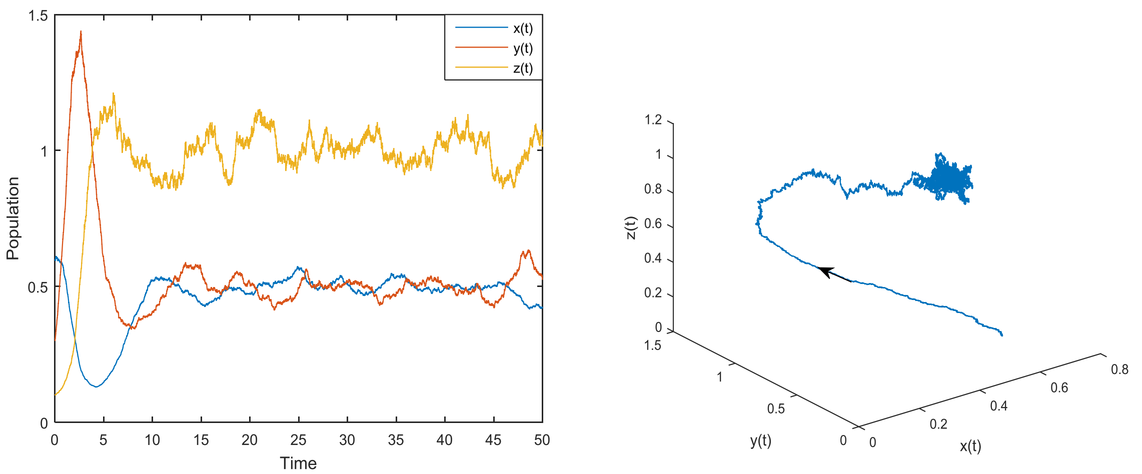

Fig.3. Time series plots along with phage portrait of system (3) with initial value $(x(0), y(0), z(0)) = (0.6, 0.3, 0.1)$and parameters are defined in (5) with σ_1= 2, σ_2 = 3, σ_3 = 4.

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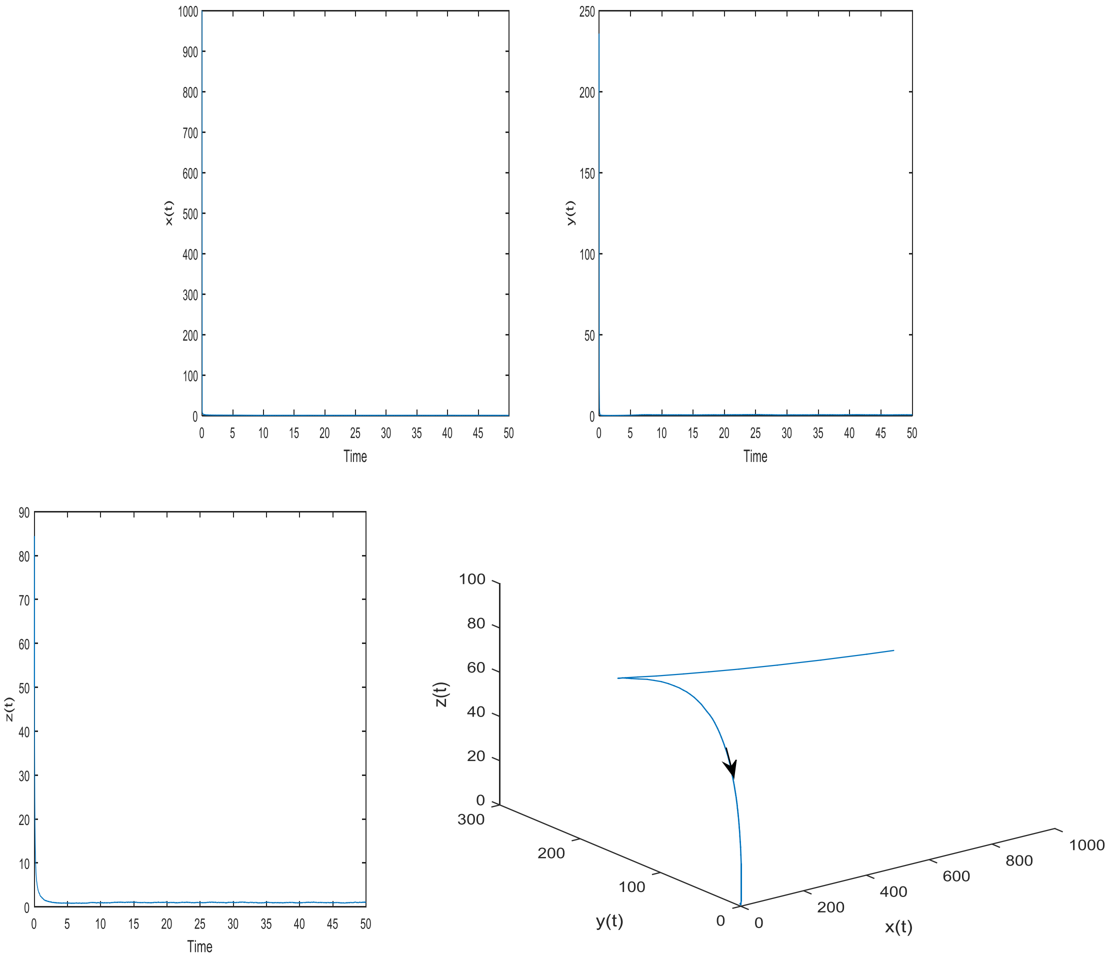

Fig.4.Time series plots of system (3) with initial value (x(0),y(0),z(0)) = (1000,200,50) and parameters are defined in (5) with noise $\sigma_1 = 2, \sigma_2 = 3, \sigma_3 = 4$.

V. DISCUSSION

In recent years the "ecology of fear" becomes an important topic regarding predator-prey interactions. Several experiments have been done on this issue. The pioneer work of Zanette et al. [20] motivated many researchers to study the dynamical behaviour of predator-prey model with fear effect.On the other hand, playback experiment in prey-predator system have tested meso-carnivores responses to large carnivores [17]. Based on the experimental study in [17], Wang and Zou [18] analysed trophic cascade in a three species food chain model with fear effect. We extend to consider and analyse the food chain model with stochastic perturbation as deterministic dynamics is already studied in [18].

To investigate the effect of environmental noise on the deterministic system, we stochastically perturb system (2) with respect to white noise which is proportional to the variables. We use the Lyapunov function and Ito's formula to examine the unique existence of global positive solutions. Ultimate boundedness of solutions is discussed. Stochastic permanence is shown under some conditions. These conditions indicate that when the intensities of noises are not large enough, the populations remain persistent. In analyzing stochastic system, we cannot find time independent equilibrium point hence investigation is required for existence of stationary distribution for stochastic system. The main reason behind it is that it plays the role of the deterministic equilibrium point and reflects stability in stochastic sense.

Conflict of interest
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