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Review on P-ADIC Integrals Involving Special Functions over Finite Fields

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Abstract. The main object of the present review paper is to review and analyze the work on p-adic integral involving special functions. We mention the identities involving special functions derived from p-adic integrals.

I. Introduction:

p-adic number : In number theory part of mathematics of the p-adic number system for any prime number p extends the ordinary arithmetic of the rational numbers in a different way from the extension of the rational number system to the real and complex number systems. The extension is achieved by an alternative interpretation of the concept of ”closeness” or absolute value. In particular, two p-adic numbers are considered to be close when their difference is divisible by a high power of p: the higher the power, the closer they are. This property enables p-adic numbers to encode congruence information in a way that turns out to have powerful applications in number theory – including, for example, in the famous proof of Fermat’s Last Theorem by Andrew Wiles. These numbers were first described by Kurt Hensel in 1897,The p-adic numbers were motivated primarily by an attempt to bring the ideas and techniques of power series methods into number theory.

p-adic expansion of rational numbers : The decimal expansion of a positive rational number r is its representation as a series

$$r = \sum_{i=k}^{\infty} a_i 10^i$$

where k is an integer and each a_i is also an integer such that $0 \leq a_i \leq 10$.

The $p - adic$ expansion of r is the formal power series

$$r = \sum_{i=k}^{\infty} a_i p^i$$

In a $p - adic$ expansion, all a_i are integers such that $0 \leq a_i < p$.

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P – adic analysis

The field of p-adic analysis, the Volkenborn integral is a method of integration for p-adic functions.

Definition 1. *Let $f : \mathbb{Z}_p \to \mathbb{C}_p$ be a function from the $p-$ adic integers taking values in the $p-$ adic numbers. The Volkenborn integral is defined by the limit, if it exists:*

$$\int_{\mathbb{Z}_p} f(x) = \lim_{n\to\infty} \frac{1}{p^n} \sum_{x=0}^{p^{n-1}} f(x)$$

More generally, if

$$R_n = \{x = \sum_{i=r}^{n-1} b^i x^i | b^i = 0, 1, 2, ..., p-1 \ for \ r < n\}$$

then

$$\int_K f(x)dx = \lim_{n\to\infty} \frac{1}{p^n} \sum_{x\in R_n\cap K} f(x)$$

This integral was defined by Arnt Volkenborn.

Examples.

$\int_{\mathbb{Z}_p} 1 \ dx = 1$
$\int_{\mathbb{Z}_p} x \ dx = -\frac{1}{2}$
$\int_{\mathbb{Z}_p} x^2 \ dx = \frac{1}{6}$
$\int_{\mathbb{Z}_p} x^k \ dx = B_k$.
where B_k is the k-th Bernoulli number.

The above four examples can be easily checked by direct use of the definition and Faulhaber's formula.

$\int_{\mathbb{Z}_p} \binom{x}{k} dx = \frac{-1^k}{k+1}$
$\int_{\mathbb{Z}_p} (1+a)^x dx = \frac{1+a}{a}$
$\int_{\mathbb{Z}_p} e^{ax} dx = \frac{a}{e^a - 1}$

The last two examples can be formally checked by expanding in the Taylor series and integrating term-wise.

$$\int_{\mathbb{Z}_p} log_p(x+u)du = \psi(x)$$

with $\log_p$ the p-adic logarithmic function and ψ_p the p-adic digamma function.

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properties

$$\int_{\mathbb{Z}_p} (x+m)dx = \int_{\mathbb{Z}_p} f(x)dx + \sum_{x=0}^{m-1} f^1(x)$$

$$if \ P^t = p^t \mathbb{Z}_p \ then \ \int_{P^t} f(x)dx = \frac{1}{p^t} \int_{\mathbb{Z}_p} f(p^t x)dx.$$

Arnt Volkenborn invented p-adic integral with respect to the following Haar measure about the end of the third quarter of the twentieth century. Despite there have been so many scientific studies for these topics since more than four decades and see also the references cited in each of these earlier works), the Volkenborn integral is today a hot topic and still keeps its mystery. Moreover, it is penetrated multifarious mathematical research areas such as special functions, the functional equations of zeta functions, number theory, Stirling numbers, Mittag-Leffler function, and Mahler theory of integration with respect to the ring Z_p in conjunction with Iwasawa's p-adic L functions. The fermionic p-adic invariant integral is firstly considered by Taekyun Kim a Korean mathematician, in order to investigate several special numbers and polynomials which can be represented by the fermionic p-adic integrals and see the references cited therein). Then, this integral has been more common, and it is used in many mathematical fields. In this study, we firstly focus on the Volkenborn integral and fermionic p-adic integral with their properties and reflections on the special polynomials and numbers. We then consider the q-extensions of the aforementioned integrals (p-adic q-integral and fermionic p-adic q-integral) and use these integrals in order to define q-analogues of the classical special polynomials and numbers

International Journal of Engineering, Management and Humanities (IJEMH)
Volume 5, Issue 1, Jan.-Feb., 2024 pp: 270-278 www.ijemh.com

p – Adic Integrals on $\mathbb{Z}_p$

We study the two-type p-adic integrals: the first is based on the Haar measure and the second is the fermionic p-adic invariant integral.Imagine that p be a fixed prime number. Throughout this part, Z_p, Q, Q_p, *and* C_p will denote the ring of p-adic rational integers, the field of rational numbers,the field of p-adic rational numbers, and the completion of algebraic closure of Q_p,respectively. Let $N = 1, 2, 3, ...$ and $N^* = N \cup 0$. The normalized absolute value according to the theory of p-adic analysis is given by $|p|_p = p^{-1}$.

The Volkenborn integral and its several generalizations have been used to introduce and research some special polynomials and numbers such as Bernoulli, Daehee polynomials and numbers.

Definition 2. *Let* $f \in C^1(\mathbb{Z}_p \to \mathbb{C}_p) = \{f|f : X \to \mathbb{C}_p \; iscontinuous\}$. *The uncertain sum of f is denoted Sf and is defined as*

$$Sf(n) = \sum_{j=0}^{n} f(j)(n \in \mathbb{N})$$

If $f \in C^1(\mathbb{Z}_p \to \mathbb{C}_p)$, *then we get* $Sf \in C^1(\mathbb{Z}_p \to \mathbb{C}_p)$.

Here we present the definition of the Volkenborn integral and its some properties

Definition 3. *Let* $f \in C^1(\mathbb{Z}_p \to \mathbb{C}_p)$ *be a function from the p-adic integers taking values in the p-adic numbers. The Volkenborn integral is defined by the limit, if it exists :*

$$\int_{\mathbb{Z}_p} f(x) d\mu(x) = \frac{1}{P^n} \lim_{N \to \infty} \sum_{x=0}^{\infty} f(x)$$

Results Let p be fixed prime. The $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$ *and* $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of rational numbers, complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$ for $x \in \mathbb{C}_p$, we use the notation $[x]_q = (1 - q^x)/(1 - q)$. The carlitz's q-bernolli numbers β_k, q can determined inductively by

$$\beta_0, q = 1, \; q(q\beta + 1)^k - \beta_k, q = \begin{cases} 1 & k = 1 \\ 0 & if \; l > 1, \end{cases}$$

International Journal of Engineering, Management and Humanities (IJEMH)
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with the usual convention of replacing β^i *by* β_i, q. (See Refs. [2, 3, 18, 21])

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients $(F_f(x,y) = (f_(x) - f_(y))/(x-y))$ have a limit $l = f'(a)$ as $(x,y) \to (a,a)$ For $f \in UD(\mathbb{Z}_p)$, let us start with expression

$$\frac{1}{[p^N]_q} \sum_{0 \le j < p^N} q^j f(j) = \sum_{0 \le j < p^N} f(j) \mu_q (j + p^N \mathbb{Z}_p)'$$

representing a q_a *analogue* of Riemann sums. The integral of f on $\mathbb{Z}_p$ will be defined as limit $(n \to \infty)$ of those sums, when it exists. The q-deformed bosonic p-adic integral of the function $f \in UD\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[dp^N]} \sum_{0 \le x < dp^N} f(x) q^x, \tag{1.1}$$

See Ref. [23] From (1.1) we note that

$$I_1(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x) dx = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N - 1} f(x). \tag{1.2}$$

In Ref. [23] It was shown that the Carlitz's q-Bernoulli numbers can be represented by p-adic q-integral on $\mathbb{Z}_p$ as follows :

$$\int_{\mathbb{Z}_p} [x]_q^m d\mu_q(x) = \beta_{m,q}, m \in \mathbb{Z}_+. \tag{1.3}$$

Thus, we have $\lim_{q \to 1} \int_{\mathbb{Z}_p} [x]_q^m d\mu_q(x) = \int_{\mathbb{Z}_p} x^n dx = B_n$, where B_n are the nth ordinary Bernoulli numbers. In the recent the p-adic invariant integral on $\mathbb{Z}_p$ are studied by several researchers in the area of number theory and mathematical physics. (See Refs. [1-22])
The p-adic norm is normally defined by $|p|_p = \frac{1}{p}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = I_1(f) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N - 1} f(x), \tag{1.4}$$

From (1.4), we can easily derive the following equation :

$$I_0(f_n) - I_0(f) = \sum_{a=0}^{n-1} f'(a), (n \ge 1), \tag{1.5}$$

where $f_n(x) = f(x+1)$, (see Refs.28,29)

In particular, $n = 1$, we have

$$I_0(f_n) - I_0(f) = f'(0) \tag{1.6}$$

From (1.5), we have

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_o(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

where $B_n(x)$ are called the bernoulli polynomials, when $x = 0$, $B_n = B_n(0)$ are called the bernoulli polynomials numbers. For $\lambda, t \in \mathbb{C}_p$ *with* $|\lambda t|_p < \frac{-1}{P^{p-1}}$, the degenerate Bernoulli polynomials are defined by carlitz to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \tag{1.7}$$

Note that $\lim_{\lambda \to 0} = B_n(x), (n \geq 0)$.

we consider the degenerate Bernoulli polynomials which are different Caritz,s degenerate

Bernoulli polynomials. These polynomials are called the degenerate bernoulli polynomials of second kind. From our degenerate polynomials, we derive some interesting ideantities and formulae related to bernoulli numbers and polynomials.

The normalized p-adic is given by $|p|_p = \frac{1}{p}$. For $f : \mathbb{Z}_p \to \mathbb{C}_p$ (f being continuous function), the fermonic p-adic integral of f is defined by kim (see Refs. 8-15, 30)

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(y) d\mu_{-1}(y) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{y=0}^{p^N - 1} f(y)(-1)^y. \tag{1.8}$$

From (1.8) follows that

$$\int_{\mathbb{Z}_p} f(y+1) d\mu_{-1}(y) + \int_{\mathbb{Z}_p} f(y) d\mu_{-1}(y) = 2f(0), \tag{1.9}$$

L.Carlitz considered the generating function for degenerate Euler polynomials by

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{m=0}^{\infty} \xi_m(\lambda|x) \frac{t^m}{n!}, \lambda \neq 0. \tag{1.10}$$

Note that, $\lim_{\lambda \to 0} \xi_m(\lambda|x) = E_m(x)$ are called classical Euler polynomials for $m \geq 0$.

Applying (1.8) with $f = (1+\lambda t)^{\frac{x+y}{\lambda}}$, we drive

$$\int_{\mathbb{Z}_p} 1 + \lambda t)^{\frac{x+y}{\lambda}} d\mu_{-1}(y) = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{m=0}^{\infty} \xi_m(\lambda|x) \frac{t^m}{n!}, \tag{1.11}$$

(see Refs. [32])

which formulates the generating function for degenerate Euler polynomial with $t, \lambda \in \mathbb{C}_p$ and $|\lambda t|_p < P^{\frac{-1}{p-1}}$.

International Journal of Engineering, Management and Humanities (IJEMH)
Volume 5, Issue 1, Jan.-Feb., 2024 pp: 270-278 www.ijemh.com

Now, using $\frac{2}{e^t+1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!}$ for $d(\in \mathbb{N})$ such that $d \equiv 1(mod2)$ we obtain

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x}{\lambda}} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} d\mu_{-1}(x) = 2\sum_{l=0}^{d-1} (1+\lambda t)^{\frac{l}{\lambda}} (-1)^l \tag{1.12}$$

Next,from (1.10) (21.11) *and* (1.12), one gets

$$\xi_n(d|\lambda) + \xi_n(\lambda) = 2\sum_{l=0}^{d-1} n! \binom{\frac{l}{\lambda}}{n} (-1)^l \lambda^n; n \geq 0, \tag{1.13}$$

where $\xi_n(\lambda) = \xi_n(0|\lambda)$ are known as degenerate Euler numbers.

We define

$$(z|\lambda)_m = z(z-\lambda)(z-2\lambda)\ldots(z-\lambda(m-1)) \; and \; (z|\lambda)_0 = 1 \tag{1.14}$$

$$Note\ that\ \lim_{\lambda\to 1} (z|\lambda)_m = z(z-1)(z-2)\ldots(z-m+1) = (z)_m; (m \geq 1).$$

Thus, from (2.0.13) *and* (2.0.14) we come to

$$\xi_n(d|\lambda) + \xi_n(\lambda) = 2\sum_{l=0}^{d-1} n!(l|\lambda)_n(-1)^l. \tag{1.15}$$

Using Volkenborn integral integral, a recent two varible extension for the degenerate Bernoulli polynomials $B_n(x|\lambda)$ is obtained by Haroon and khan as [31]

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+x_1}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} d\mu_0(x_1) = \frac{log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} \tag{1.16}$$
$$= \sum_{m=0}^{\infty} {}_H B_m(x,y|\lambda) \frac{t^m}{m!},$$

where the generating function in the r.h.s of (2.0.16) is evaluated by appling (2.0.8) to the integral in l.h.s.

The generalization in (2.0.16) can be witnessed due to carlitz' degenerate Bernoulli polynomials $\beta_n(x,\lambda)$ (see[2]) and kampe de Feriet generalization of Hermite polynomials $H_n(x,y)$ for two variables. Further as $\lim_{\lambda\to 1}$ in (2.0.16), the result of Dattoli et al. involving Hermite-Bernoulli polynomials can be seen :

$$(\frac{t}{e^t-1})e^{xt+yt^2} = \sum_{m=0}^{\infty} {}_H B_m(x,y) \frac{t^m}{m!}. \tag{1.17}$$

p − adic integrals involving special functions :

Degenerate Bernoulli Numbers and Polynomials of the Second Kind :

Kim and Seo [24] constructed degenerate Bernoulli numbers and polynomials which are slightly different carlitz's degenerate Bernoulli numbers and polynomials. From our degenerate Bernoulli numbers and polynomials, we derive same identities and formulate related to Bernoulli numbers and polynomials .

On p-adic Integral for degenerate Hermite-Euler numbers and polynomials:

Khan at.al [25] studied the degenerate Hermit-Euler polynomials arising from p-adic integrals on $\mathbb{Z}_P$ Further, we extend several identities established by kim. Some ideantities

International Journal of Engineering, Management and Humanities (IJEMH)
Volume 5, Issue 1, Jan.-Feb., 2024 pp: 270-278 www.ijemh.com

of symmetry involving these polynomials are also derived .

On p-adic integral for generalized degenerate Hermite-Bernoulli polynomials attached to of higher order :

Khan and Haroon, [26] In the current investigation, Obtained the generating function for Hermite-based degenerate Bernoulli polynomials attached to ξ of higher order using p-adic methods over the ring of integers. Useful indeantities, formulae and relations with well known families of polynomials and numbers including the Bernoulli numbers, Daehee numbers and the stirling numbers are established. We also give identities of symmetry and additive property for Hermite-based generalized degenerate Bernoulli polynomilas attached to ξ of higher order. Results are supported by remarks and corollaries .

Conclusion :

The Volkenborn integral was firstly introduced by, a German mathematician, Arnt Volkenborn, see Ref [33] and [34] With the introduction of the mentioned integral, this integral has been utilized to define and investigate several special polynomials and numbers such as Daehee and Bernoulli numbers and polynomials along with their diverse generalizations. The fermionic p-adic integral is firstly introduced by Korean mathematician Taekyun Kim in 2005. In conjunction with the introduction of this integral, the foregoing integral and its many extensions have been used to consider and analyze several special numbers and polynomials such as Euler, Genocchi, Frobenius-Euler, Eulerian, Changhee, Boole and their many extensions polynomials and numbers. Nowadays, the Volkenborn integral and the fermionic p-adic integral have been more common, and they are used in multifarious mathematical research areas and physical research are

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